Question 1 **(7 marks)**

(a) Write $\frac{1+i\sqrt{3}}{1+i}$ in the form $x + yi$, where x and y are real numbers. (2 marks)

$$\frac{(1+i\sqrt{3})(1-i)}{(1+i)(1-i)} = \frac{1 - i + i\sqrt{3} + \sqrt{3}}{2}$$

$$= \left(\frac{1+\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}-1}{2}\right)i$$

✓ Multiply by $\frac{(1-i)}{(1-i)}$

✓ Correct Answer

(b) By expressing both $1 + i\sqrt{3}$ and $1 + i$ in polar form $r \operatorname{cis} \theta$, show that $\frac{1+i\sqrt{3}}{1+i} = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right)\right)$ (3 marks)

$$1+i\sqrt{3} = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$$ ✓ To polar form

$$1+i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$ ✓ To polar form

$$\therefore \frac{1+i\sqrt{3}}{1+i} = \frac{2\operatorname{cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)}$$

$$= \sqrt{2}\operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

✓ Correct working out

$$= \sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)$$

(c) Hence, using your answers from parts (a) and (b), find the exact value of $\sin\left(\frac{\pi}{12}\right)$. (2 marks)

$$\sqrt{2}\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2}$$

✓ Equate imaginary parts

$$\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

✓ correct simplification

Question 2 **(8 marks)**

(a) Show that $(1+i)^5 = -4-4i$. (3 marks)

$(1+i)^2 = 2i$, so

$(1+i)^5 = (2i)(2i)(1+i)$ ✓ Expands $(1+i)^5$

$= -4(1+i)$ ✓ Shows one correct step

$= -4-4i$ ✓ shows all correct steps

(b) Hence determine all the roots of the equation $z^5 = -4-4i$, expressing each in the form $r \text{ cis } \theta$, with $r \geq 0$ and $-180° < \theta \leq 180°$. (3 marks)

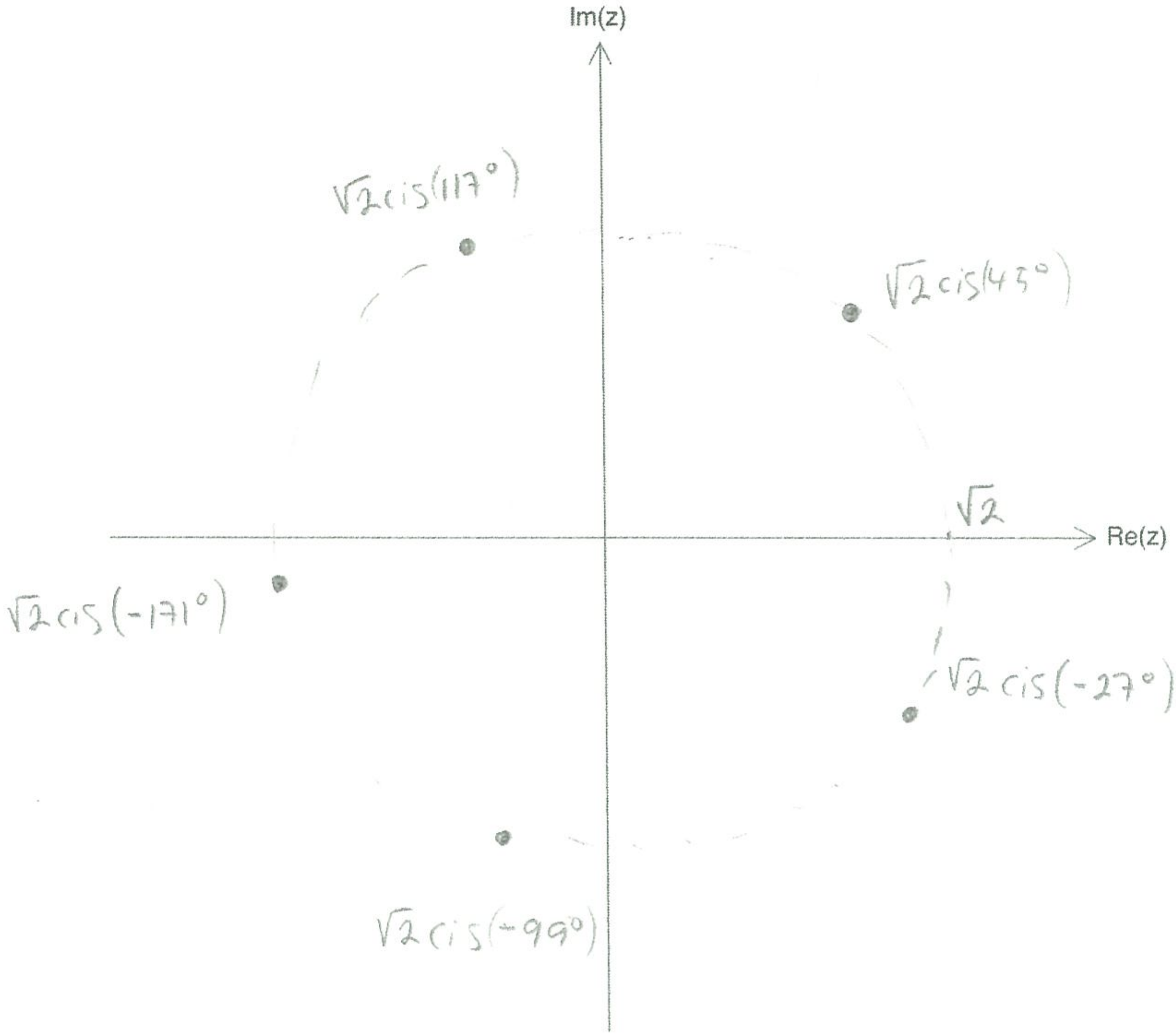
$1+i = \sqrt{2} \text{ cis}(45°)$ ✓ for $\sqrt{2}$ ✓ for 45°

$\frac{360°}{5} = 72°$

$$\therefore \quad z = \begin{cases} \sqrt{2} \text{ cis}(45°) \\ \sqrt{2} \text{ cis}(117°) \\ \sqrt{2} \text{ cis}(-27°) \\ \sqrt{2} \text{ cis}(-99°) \\ \sqrt{2} \text{ cis}(-171°) \end{cases}$$

✓ all five correct

(c) Sketch the roots from part (b) in the complex plane below. (2 marks)



✓ On a circle shape

✓ Evenly spread

Question 3 **(7 marks)**

De Moivre's Theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\,sin(n\theta), \text{ for any integer } n.$$

(a) Prove de Moivre's Theorem for **positive** integers. (4 marks)

To prove: $(\text{cis}\theta)^n = \text{cis}(n\theta)$, $n = 1, 2, 3, \ldots$

$n = 1$: Trivially true. ✓ $n=1$

$n = 2$: $(\text{cis}\theta)^2 = \text{cis}\theta\,\text{cis}\theta$ ✓ uses $\text{cis}\varphi\,\text{cis}\theta = \text{cis}(\varphi+\theta)$

$= \text{cis}(\theta + \theta)$

$= \text{cis}(2\theta)$, true. ✓ $n=2$

$n = 3$: $(\text{cis}\theta)^3 = (\text{cis}\theta)^2\,\text{cis}\theta$

$= \text{cis}(2\theta)\,\text{cis}\theta$ ✓ Convincing reasoning

$= \text{cis}(2\theta + \theta)$

$= \text{cis}(3\theta)$, true.

And so on...

(b) Prove de Moivre's Theorem for **negative** integers. (3 marks)

$n = -1$: $(\text{cis}\theta)^{-1} = \dfrac{1}{\text{cis}\theta}$ ✓ $n=-1$

$= \dfrac{\text{cis}(-\theta)}{\text{cis}(\theta)\,\text{cis}(-\theta)}$

$= \text{cis}(-\theta)$, true.

$n = -2$: $(\text{cis}\theta)^{-2} = \dfrac{1}{(\text{cis}\theta)^2}$ ✓ $n=-2$

$= \dfrac{\text{cis}(-2\theta)}{\text{cis}(2\theta)\,\text{cis}(-2\theta)}$

$= \text{cis}(-2\theta)$, true.

And so on ... ✓ convincing reasoning

Question 4 **(6 marks)**

(a) Expand $(\cos\theta + i\sin\theta)^5$ and write your answer in the form $a + ib$. (2 marks)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$1\cos^5\theta(i\sin\theta)^0 + 5\cos^4\theta(i\sin\theta)^1 + 10\cos^3\theta(i\sin\theta)^2$$
$$+ 10\cos^2\theta(i\sin\theta)^3 + 5\cos^1\theta(i\sin\theta)^4 + 1(\cos\theta)^0(i\sin\theta)^5$$

✓ correct expanding

$$= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right)$$
$$+ i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)$$

✓ correct grouping

(b) Use de Moivre's Theorem and your result from part (a) to show that $\sin(5\theta) = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. (4 marks)

$$\sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5\left((1-\sin^2\theta)^2\right)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(\sin^3\theta - \sin^5\theta) + \sin^5\theta$$
$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$
$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

✓ replace $\cos^2\theta$ with $(1-\sin^2\theta)$

✓ replace $\cos^4\theta$ with $(1-\sin^2\theta)^2$

✓ expands correctly

✓ arrives successfully at the answer

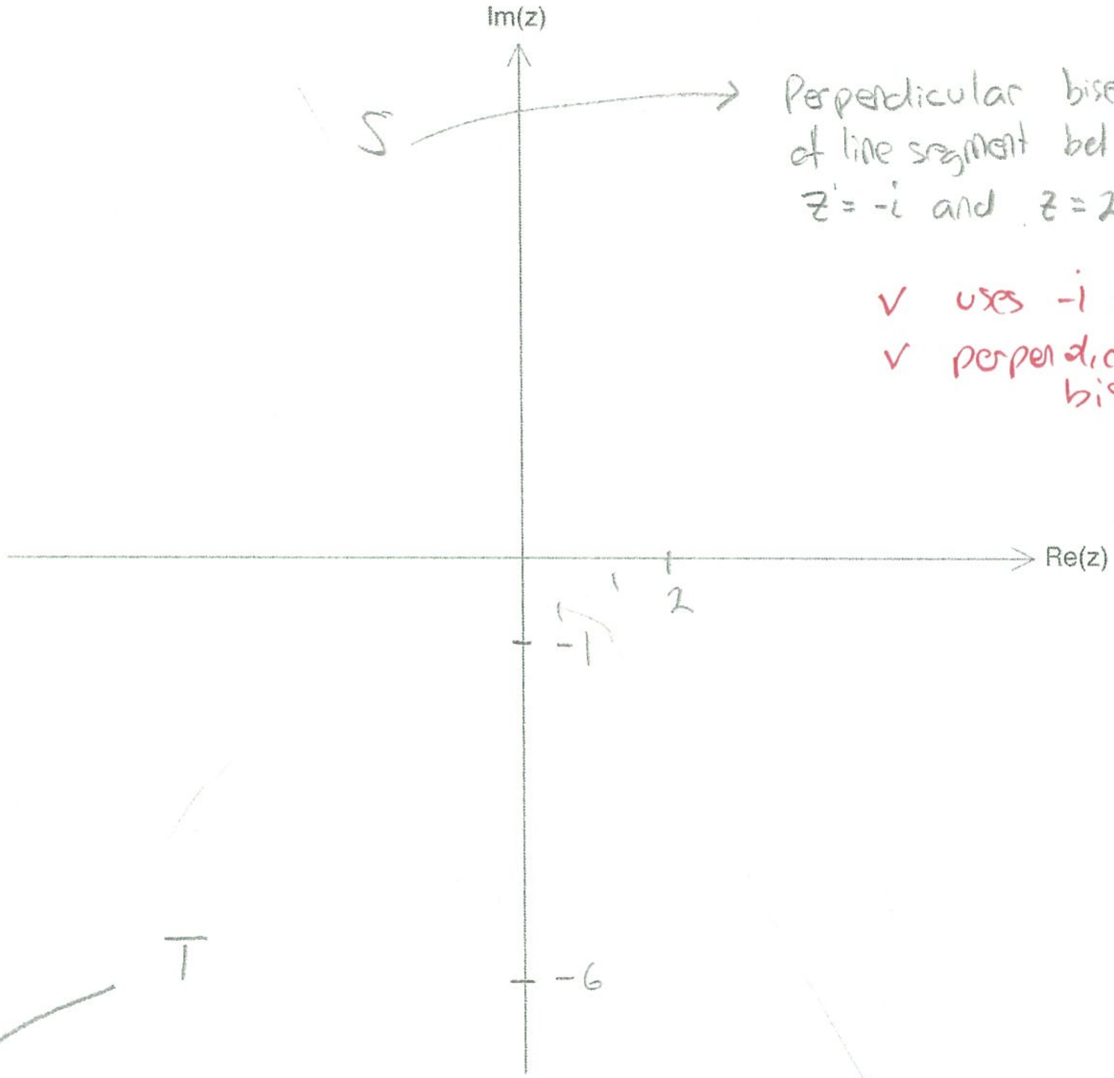
Question 5 **(5 marks)**

Consider the following sets of complex numbers:

$$S = \{z : |z + i| = |z - 2|\}$$

$$T = \left\{z : \left|\frac{z - 2i}{z + 2i}\right| = \sqrt{2}\right\}$$

Sketch the two sets of complex numbers in the Argand diagram below.



Perpendicular bisector of line segment between $z = -i$ and $z = 2$.

✓ uses $-i$ and 2
✓ perpendicular bisector

$|z - 2i| = \sqrt{2}\,|z + 2i|$

$x^2 + (y-2)^2 = 2(x^2 + (y+2)^2)$

$x^2 + y^2 - 4y + 4 = 2x^2 + 2y^2 + 8y + 8$

$\vdots$

$32 = x^2 + (y+6)^2$

$\therefore$ a circle with centre at $(0, -6)$ and radius $\sqrt{32}$.

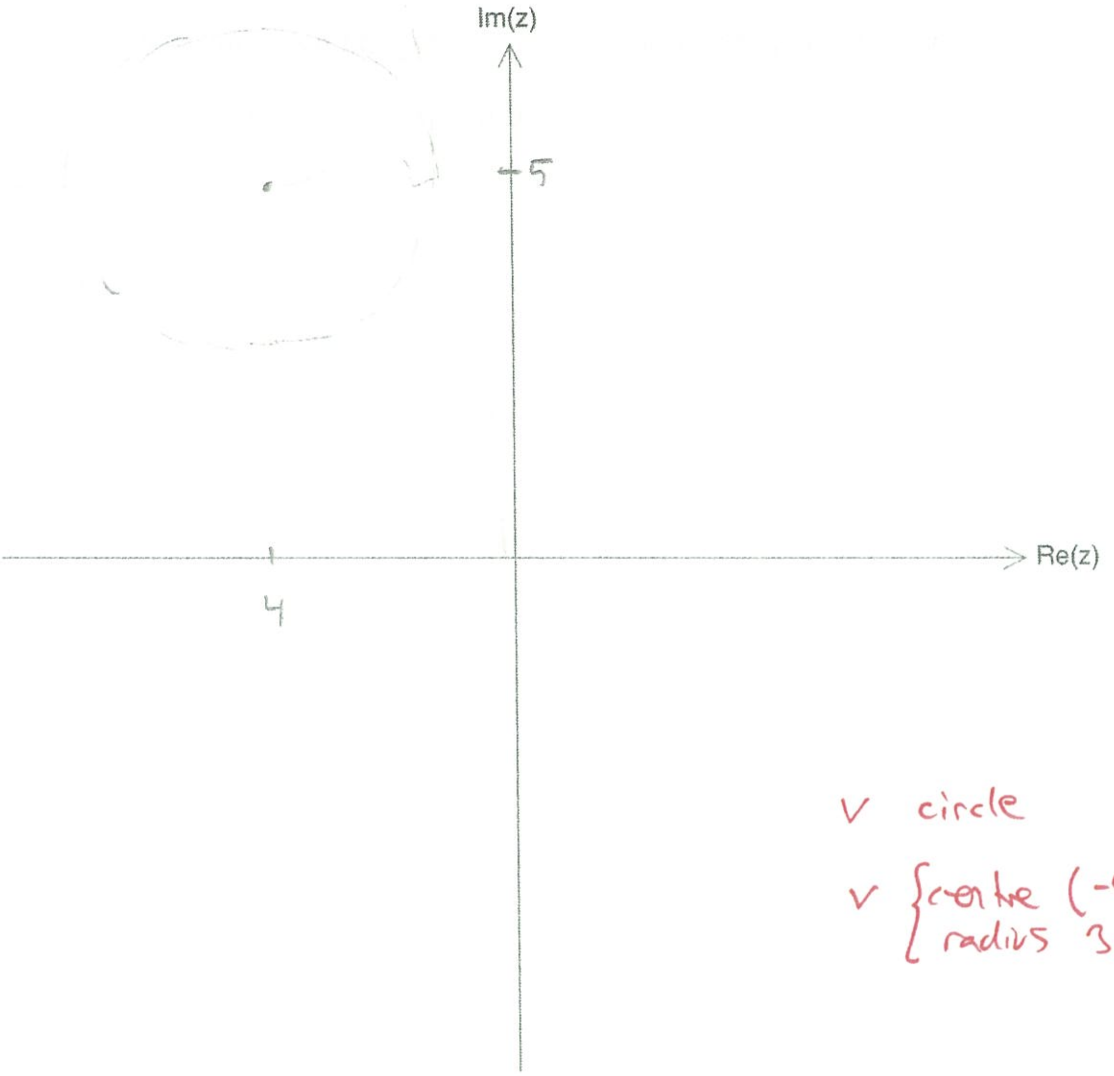
✓ uses $z = x + iy$
✓ $|z - 2i| = \sqrt{2}\,|z + 2i|$
✓ correct ~~answer~~ circle

Question 6 **(7 marks)**

Consider the following set of complex numbers:

$$S = \{z : |z + 4 - 5i| = 3\}.$$

(a) Sketch the set of complex numbers S in the Argand diagram below. (2 marks)



✓ circle

✓ { centre $(-4,5)$
radius 3

$|z - (-4+5i)| = 3$ is a circle with centre at $(-4,5)$ and radius 3.

(b) For z in S, determine the **maximum** value of $|z|$, the modulus of z. (2 marks)

$3 + \sqrt{4^2 + 5^2}$

$= 3 + \sqrt{41}$

≈ 9.40

✓ Two parts: radius + distance to centre.

✓ correct answer

(c) For z in S, determine the **minimum** value of $\arg(z)$, the argument of z, where $-\pi < \arg(z) \le \pi$. (3 marks)

$\frac{\pi}{2} + \tan^{-1}\left(\frac{4}{5}\right) - \sin^{-1}\left(\frac{3}{\sqrt{41}}\right)$

≈ 1.76

✓ $\tan^{-1}\left(\frac{4}{5}\right)$

✓ $\sin^{-1}\left(\frac{3}{\sqrt{41}}\right)$

✓ correct answer

Question 7 **(5 marks)**

Show that, for every positive integer n, $(1+i)^n + (1-i)^n = 2(\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right)$.

$$LHS = (1+i)^n + (1-i)^n$$

$$= \left(\sqrt{2}\,\text{cis}\,\frac{\pi}{4}\right)^n + \left(\sqrt{2}\,\text{cis}\,\frac{-\pi}{4}\right)^n$$

✓ polar form

$$= (\sqrt{2})^n\,\text{cis}\left(\frac{n\pi}{4}\right) + (\sqrt{2})^n\,\text{cis}\left(\frac{-n\pi}{4}\right)$$

✓ Uses de Moivre's Theorem

$$= (\sqrt{2})^n\cos\left(\frac{n\pi}{4}\right) + i(\sqrt{2})^n\sin\left(\frac{n\pi}{4}\right) + (\sqrt{2})^n\cos\left(\frac{-n\pi}{4}\right) + i(\sqrt{2})^n\sin\left(\frac{-n\pi}{4}\right)$$

✓ cos even

✓ sin odd

$$= 2(\sqrt{2})^n\cos\left(\frac{n\pi}{4}\right)$$

✓ all steps correct

since $\cos(x) = \cos(-x)$ (even function)

and $\sin(x) = -\sin(x)$ (odd function)

End of questions